A note on the rings with flat injective hulls

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November 12, 2015

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We briefly say that F is the flat cover of M and denote it by $\mathcal{F}(M)$.

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Definition.

- A module M is Gorenstein flat if there is an exact sequence

$$\cdots \to F^{-2} \to F^{-1} \to F^{0} \to F^{1} \to F^{2} \to \cdots$$

of flat modules such that $M = ker(F^0 \to F^1)$ and such that $E \otimes_R -$ leaves the sequence exact when E is injective.

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- The Gorenstein flat dimension of M is denoted by $\mathrm{Gfd}(M)$ and defined as

$$\mathrm{Gfd}(M) = \inf\{n | \text{there exists an exact sequence}$$

$$0 \to G_n \to \cdots \to G_0 \to M \to 0 \text{ s.th. } G_i s \text{ are Gorenstein flat}\}.$$

Definition.

Let R be a ring and let $\mathcal{GF}$ be the class of all Gorenstein flat R-modules. Then for an R-module M, a morphism $\varphi : G \to M$, where $G \in \mathcal{GF}$ is called a Gorenstein flat cover of M if

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Remark.

Note that the existence of a flat cover, an injective envelope and a Gorenstein flat cover for any module over any associative ring has been proved.

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Definitions [Enochs, E. E.; Huang, Z., 2012].

Let R be a ring and n a positive integer.

- A module M is called n-cotorsion if $\mathrm{Ext}^1_R(X, M) = 0$ for any R-module X with flat dimension at most n.

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Theorem [Enochs, 1984].

Every flat module F can be uniquely written in the form

$$F = \prod T_{\mathfrak{p}},$$

where $T_{\mathfrak{p}}$ is a completion of a free $R_{\mathfrak{p}}$-module with respect to $\mathfrak{p}$-adic topology.

Definition.

A prime ideal $\mathfrak{p}$ of R is said to be a coassociated prime of M if there exists an Artinian homomorphic image L of M with $\mathfrak{p} = 0 :_R L$. The set of all coassociated prime ideals of M is denoted by $\mathrm{Coass}(M)$.

Proposition.

Let F be a flat R-module and E be an injective R-module, where R is a commutative ring with non-zero identity.

- The following conditions are equivalent.
  1. $\mathcal{E}(F)$ is flat.
  2. $\mathcal{F}(\mathcal{E}(F)) = \mathcal{E}(F)$.
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- A flat module over a Noetherian ring R is injective if and only if it is Gorenstein injective.

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- An injective module over a commutative ring R is flat if and only if it is Gorenstein flat.

Proposition.

Assume that for all injective R-modules E and E' such that $\mathrm{Ass}(E) \subseteq \mathrm{Ass}(R)$, the R-module $Hom_R(E, E')$ is injective. Then

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- $\mathcal{F}(\mathcal{E}(R))$ is injective; and
- $R_{\mathfrak{p}}$ is a Gorenstein ring of Krull dimension 0 for all $\mathfrak{p} \in \mathrm{Ass}(R)$.

Proposition.

- For each injective R-module E we have

$$\mathrm{id}_R(\mathcal{F}(E)) \leq \mathrm{fd}_R(\mathcal{E}(R)).$$

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- For each prime ideal $\mathfrak{p}$ of R,

$$\mathrm{id}_R \widehat{R_{\mathfrak{p}}} = \mathrm{fd}_R(\mathcal{E}(R/\mathfrak{p})).$$

- It is well-known that every Gorenstein ring has flat injective hull. [Bass, H., 1963].

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- There exists an example of a ring with flat injective hull which is not Gorenstein. [Enochs, E. E.; Huang, Z., 2012]
- So, the rings with flat injective hulls are generalizations of Gorenstein rings.

Theorem [Cheatham, T.; Enochs, E. E., 1980] and [Enochs, E. E., 2000].

The following are equivalent for a commutative Noetherian ring R.

1. $\mathcal{E}(R)$ is flat.
2. $R_{\mathfrak{p}}$ is a Gorenstein ring of Krull dimension 0 for all $\mathfrak{p} \in \mathrm{Ass}(R)$.
3. $\mathcal{E}(F)$ is flat for all flat R-modules F.
4. $\mathcal{F}(E)$ is injective for all injective R-modules E.
5. $E \otimes E'$ is an injective module for all injective R-modules E and E'.
6. $S^{-1}R$ is an injective R-module where S is the set of non-zero divisors of R.

Theorem [Khashyarmanesh, K.; Salarian, Sh., 2003].

The following are equivalent for a commutative Noetherian ring R.

1. $\mathcal{E}(R)$ is flat.
2. $\mathcal{E}(R)$ has finite flat dimension.
3. $\mathcal{F}(M)$ is injective for any strongly cotorsion module M.
4. $\mathcal{E}(M)$ is flat for any strongly torsion free module M.
5. $\mathcal{E}(M)$ is flat for any Gorenstein flat module M.
6. If $\mathfrak{p} \in \mathrm{Coass}(E)$ for an injective R-module E, then $\widehat{R_{\mathfrak{p}}}$ is injective.

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The following are equivalent for a commutative Noetherian ring R.

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3. $\mathcal{F}(M)$ is injective for any strongly cotorsion module M.
4. $\mathcal{E}(M)$ is flat for any strongly torsion free module M.
5. $\mathcal{E}(M)$ is flat for any Gorenstein flat module M.
6. If $\mathfrak{p} \in \mathrm{Coass}(E)$ for an injective R-module E, then $\widehat{R_{\mathfrak{p}}}$ is injective.
   If moreover the Krull dimension of R is finite, then the above conditions are equivalent to
7. $\mathcal{F}(M)$ is injective for any Gorenstein injective module M.

Theorem [Enochs, E. E.; Huang, Z., 2012].

For a Commutative Noetherian ring R the following conditions are equivalent.

(1) $\mathcal{E}(R)$ is flat.

(2) $\mathcal{E}(R)$ is Gorenstein flat.

(3) $\mathcal{E}(F)$ is Gorenstein flat for any flat R-module F.

(4) $\mathcal{E}(G)$ is Gorenstein flat for any Gorenstein flat R-module G.

(5) $\mathcal{GF}(M)$ is injective for any 1-Gorenstein cotorsion R-module M.

(6) $\mathcal{GF}(M)$ is injective for any strongly Gorenstein cotorsion R-module M.

(7) $\mathcal{GF}(E)$ is injective for any injective left R-module E.

(8) $\mathcal{E}(N)$ is flat for any 1-Gorenstein torsionfree R-module N.

(9) $\mathcal{E}(N)$ is Gorenstein flat for any 1-Gorenstein torsionfree R-module N.

(10) $\mathcal{E}(N)$ is flat for any strongly Gorenstein torsionfree R-module N.

(11) $\mathcal{E}(N)$ is Gorenstein flat for any strongly Gorenstein torsionfree R-module N.

(12) $\mathcal{F}(M)$ is injective for any 1-cotorsion R-module M.

(13) $\mathcal{E}(N)$ is flat for any 1-torsionfree R-module N.

(14) $\mathcal{E}(N)$ is Gorenstein flat for any 1-torsionfree R-module N.

(15) $\mathcal{E}(N)$ is Gorenstein flat for any strongly torsionfree R-module N.

Theorem.

For a commutative Noetherian ring R, the following conditions are equivalent.

(1) $\mathcal{E}(R)$ is flat.

(2) $\mathcal{F}(\mathcal{E}(R))$ is injective.

(3) $\mathcal{F}(\mathcal{E}(R))$ is Gorenstein injective.

(4) $\mathcal{E}(R/\mathfrak{p})$ is flat for any associated prime ideal $\mathfrak{p}$ of R.

(5) $T_{\mathfrak{p}}$ is injective for any coassociated prime ideal $\mathfrak{p}$ of $\mathcal{F}(\mathcal{E}(R))$, where $T_{\mathfrak{p}}$ is the completion of a free $\widehat{R_{\mathfrak{p}}}$-module.

(6) $\mathcal{E}(R/\mathfrak{p})$ is Gorenstein flat for any associated prime ideal $\mathfrak{p}$ of R.

(7) $T_{\mathfrak{p}}$ is Gorenstein injective for any coassociated prime ideal $\mathfrak{p}$ of $\mathcal{F}(\mathcal{E}(R))$.

(8) $\mathcal{E}(R/\mathfrak{p})$ has finite flat dimension for any associated prime ideal $\mathfrak{p}$ of R.

(9) $\mathcal{F}(\mathcal{E}(F))$ is injective for all flat R-modules F.

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(12) $\mathcal{E}(\mathcal{F}(E))$ is Gorenstein flat for all injective R-modules E.

(13) $\mathcal{F}(E)$ is Gorenstein injective for all injective R-modules E.

(14) $\mathcal{F}(M)$ is Gorenstein injective for all strongly cotorsion R-modules M.

(15) $\mathcal{E}(F)$ has finite flat dimension for all flat R-modules F.

(16) $\mathcal{E}(M)$ is flat for all R-modules M with $\mathrm{Ass}(M) \subseteq \mathrm{Ass}(R)$.

(17) $\mathcal{E}(M)$ is Gorenstein flat for all R-modules M with $\mathrm{Ass}(M) \subseteq \mathrm{Ass}(R)$.

(18) $\mathcal{E}(M)$ has finite flat dimension for all R-modules M with $\mathrm{Ass}(M) \subseteq \mathrm{Ass}(R)$.

(19) $R_{\mathfrak{p}}$ is injective for all coassociated prime ideals $\mathfrak{p}$ of $\mathcal{F}(\mathcal{E}(R))$.

(20) There is an injective R-module E such that for all $\mathfrak{p} \in \mathrm{Coass}(E)$, $\widehat{R_{\mathfrak{p}}}$ is injective.

(21) For all injective R-modules E and E' the R-module $E \otimes_R E'$ is injective and flat.

(22) For all injective R-modules E and E' such that $\mathrm{Ass}(E) \subseteq \mathrm{Ass}(R)$, the R-module $Hom_R(E, E')$ is injective and flat.

If moreover the Krull dimension of R is finite, the above conditions are equivalent to:

(23) $\mathcal{F}(M)$ is Gorenstein injective for all Gorenstein injective R-modules M.

Also, if every prime ideal in $\mathrm{Ass}(R)$ is a minimal prime ideal of R, then the condition "$\mathcal{E}(R)$ is flat" is equivalent to the following conditions.

(24) $\mathcal{F}(\mathcal{E}(R))$ has finite injective dimension.

(25) $T_{\mathfrak{p}}$ has finite injective dimension for any coassociated prime ideal $\mathfrak{p}$ of $\mathcal{F}(\mathcal{E}(R))$.

(26) $\mathcal{F}(E)$ has finite injective dimension for all injective R-modules E.

(27) Every flat and cotorsion R-module F such that $\mathrm{Coass}(F) \subseteq \mathrm{Ass}(R)$ is injective.

(28) Every flat and cotorsion R-module F such that $\mathrm{Coass}(F) \subseteq \mathrm{Ass}(R)$ is Gorenstein injective.

(29) $R_{\mathfrak{p}}$ is Gorenstein for all coassociated prime ideals $\mathfrak{p}$ of $\mathcal{F}(\mathcal{E}(R))$.

Thanks for your patience.